Maximal L_p-Regularity and R-sectorial Operators (Nonlinear evolution equations and applications)

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1 R-boundedness and operator-valued multiplier theorems

Definition 1.1 *Let $(X, \|\cdot\|)$ be a real or complex Banach space. A collection $\mathcal{T}$ of bounded linear operators in X is said to be R-bounded (Randomized bounded) if there exists a constant $M \geq 0$ such that*

$$\sum_{\varepsilon\in\{-1,1\}^N} \|\sum_{k=1}^{N} \varepsilon_k T_k x_k\|^2 \leq M^2 \sum_{\varepsilon\in\{-1,1\}^N} \|\sum_{k=1}^{N} \varepsilon_k x_k\|^2 \tag{1.1}$$

holds for all $\{T_k\}_{k=1}^N \subset \mathcal{T}$, all $\{x_k\}_{k=1}^N \subset X$ and all $N = 1, 2, \ldots$. A constant $M \geq 0$ such that (1.1) holds is called an R-bound for $\mathcal{T}$ and the smallest one is denoted by $\mathcal{R}_2(\mathcal{T})$. (If the collection $\mathcal{T}$ is not R-bounded we set $R_2(\mathcal{T}) = \infty$).

The first explicit definition of R-boundedness can be found in [2], although this notion was already used by J. Bourgain in [3]. For a systematic treatment of this notion see [4], [15].

Remarks

1. The notion of R-boundedness can be trivially extended to a family of operators acting from a Banach space X into a Banach space Y.

2. By taking $N = 1$ in (1.1) one finds that a R-bounded collection of operators is uniformly bounded.

3. A finite collection of bounded operators is R-bounded.

4. By using the generalized parallelogram law

$$\sum_{k=1}^{N} \|x_k\|^2 = \frac{1}{2^N} \sum_{\varepsilon \in \{-1,1\}^N} \|\sum_{k=1}^{N} \varepsilon_k x_k\|^2 \quad ; \tag{1.2}$$

one shows that in a Hilbert space, the notion of R-boundedness and uniform boundedness are equivalent.

5. The RHS of (1.2) can be rewritten as $\| \sum_{k=0}^{N} \varepsilon_k x_k \|^2_{L_2(\Omega;X)}$, where $\{\varepsilon_k\}_{k-1}^{\infty}$, denotes a sequence of independent identically distributed symmetric $\{-1,1\}$-valued random variables defined on some probability space (Ω, F, P). Accordingly condition (1.1) can be rewritten as

$$\| \sum_{n-1}^{N} \varepsilon_k T_k x_k \|_{L_2(\Omega;X)} \leq M \| \sum_{k=1}^{N} \varepsilon_k x_k \|_{L_2(\Omega;X)} \quad . \tag{1.3}$$

In view of Kahane's inequality we can replace $L_2(\Omega; X)$ by $L_p(\Omega; X)$, $1 \leq p < \infty$, adjusting the constant M appropriately.

It appears that the notion of R-boundedness is useful in the context of multiplier theorems associated with unconditional Schauder decompositions.

Definition 1.2 *A sequence $D = \{D_k\}_{k-1}^{\infty}$ of bounded linear projections in X is called a Schauder decomposition of X, if*

$$D_k D_l = 0 \quad \textit{whenever} \quad k \neq l \quad , \tag{1.4}$$

$$x = \sum_{k=1}^{\infty} D_k x \quad \textit{for all} \quad x \in X \quad . \tag{1.5}$$

The decomposition is called unconditional if the series in (1.5) is unconditionally convergent for all $x \in X$.

We recall that if D is an unconditional Schauder decomposition of X, then there exists a constant $c_D > 0$ such that

$$c_D^{-1}\|\sum_{k=1}^{n} D_k x\| \leq \|\sum_{k=1}^{n} \varepsilon_k D_k x\|_{L_2(\Omega;X)} \leq c_D\|\sum_{k=1}^{n} D_k x\| \quad , \tag{1.6}$$

holds for all $x \in X$ and all $n \geq 1$.

Let $D = \{D_k\}_{k=1}^{\infty}$ be a Schauder decomposition of X and let $L = \{L_k\}_{k=1}^{\infty}$ be a sequence of bounded linear operators such that L_k leaves $R(D_k)$, the range of D_k, invariant, for each $k \geq 1$, i.e.

$$L_k D_k = D_k L_k D_k, \quad k \geq 1 \quad . \tag{1.7}$$

Let X_D be the linear subspace of X generated by the subspaces $\{R(D_k)\}_{k=1}^{\infty}$, i.e.

$$X_D := \bigcup_{k-1}^{\infty} R(\sum_{l=1}^{k} D_l) \quad . \tag{1.8}$$

We observe that X_D is dense in X and that X_D is invariant under D_k, $k \geq 1$. We still denote the restriction of D_k to X_D by D_k and define

$$T_L x := \sum_{k=1}^{\infty} L_k D_k x \quad , \quad \text{(finite sum)} \quad , \tag{1.9}$$

for all $x \in X_D$. The operator T_L maps X_D into itself and commutes with D_k:

$$T_L D_k = D_k T_L \quad , \quad k \geq 1 \quad . \tag{1.10}$$

In case the decomposition D is unconditianal, we obtain from (1.6), (1.9) and (1.10):

$$c_D^{-1}\|T_L x\| \leq \|\sum_{k=1}^{\infty} \varepsilon_k L_k D_k x\|_{L_2(\Omega;X)} \leq c_D\|T_L x\| \quad , \tag{1.11}$$

for all $x \in X_D$. Since X_D is dense in X and X is complete, the operator T_L extends to a bounded linear operator on X iff T_L is bounded on X_L. In view of (1.11) we have the following characterization.

Theorem 1.1 *([4], [15]). Let $D = \{D_k\}_{k=1}^{\infty}$, be an unconditional Schauder decomposition of the Banach space X and let $L = \{L_k\}_{k=1}^{\infty}$, be a sequence of bounded linear operators of X satisfing (1.7).*

Let X_D be the dense linear subspace of X defined by (1.8) and let T_L be the linear operator on X_L defined by (1.9).

Then the operator T_L is bounded iff there exists a constant $M > 0$ such that

$$\|\sum_{k=1}^{n} \varepsilon_k L_k x_k\|_{L_2(\Omega;X)} \leq M\|\sum_{k=1}^{n} \varepsilon_k x_k\|_{L_2(\Omega;X)} \tag{1.12}$$

holds, for all $x_k \in R(D_k)$, $k \geq 1$ and all $n \geq 1$.

If this condition id fulfilled then the norm of T_L satisfies

$$\|T_L\| \leq c_D^2 M \quad . \tag{1.13}$$

In particular if the collection L is R-bounded with constant $R_2(L)$, then (1.12) holds with $M = R_2(L)$.

Proof: If (1.12) holds and $x = \sum_{k=1}^{n} D_k x$ for some $n \geq 1$ we have $\|T_L x\| \leq c_D\|\sum_{k=1}^{n} \varepsilon_k L_k D_k x\|_{L_2(\Omega;X)} \leq c_D M\|\sum_{k=1}^{n} \varepsilon_k D_k x\|_{L_2(\Omega;X)} \leq c_D^2 M\|\sum_{k-1}^{n} D_k x\| = c_D^2 M\|x\|$, where we have used (1.11) and (1.6).

Conversely if T_L is bounded, we have for all $x_k \in R(D_k),\ k = 1, ..., n$ and all $n \geq 1$,

$$\begin{aligned}
&\|\sum_{k=1}^{n} \varepsilon_k L_k x_k\|_{L_2(\Omega;X)} = \\
&\|\sum_{k=1}^{n} \varepsilon_k L_k D_k x\|_{L_2(\Omega;X)} \leq c_D \|T_L x\| \\
&\leq c_D \|T_L\|\ \|x\| = c_D \|T_L\|\ \|\sum_{k=1}^{n} D_k x\| \\
&c_D^2 \|T_L\|\ \|\sum_{k=1}^{n} \varepsilon_k D_k x\|_{L_2(\Omega;X)} = \\
&c_D^2 \|T_L\|\ \|\sum_{k=1}^{n}\|_{L_2(\Omega;X)} \quad ,
\end{aligned}$$

where $x = \sum_{k=1}^{n} D_k x_k$ and with the use of (1.6), (1.11).

2 Operator-valued Marcinkiewicz and Mikhlin multiplier theorems

Let $(X, \|\cdot\|)$ be a complex Banach space and let $L^p(0,1;X),\ 1 \leq p < \infty$ denote the usual complex Banach space equipped with the norm

$$\|u\|_p := (\int_0^1 \|u(t)\|^p dt)^{1/p} \quad .$$

Let $N = 0, 1, 2, ...$ and let

$$\begin{aligned}
F_N u &:= \sum_{k=-N}^{N} e_k \otimes \hat{u}(k), \quad \text{where} \quad u \in L^p(0,1;X) \quad , \\
e_k(t) &:= e^{2\pi k i t}; \quad t \in [0,1], \quad k \in \mathbb{Z} \quad , \\
\hat{u}(k) &:= \int_0^1 e^{-2\pi k i t} u(t) dt, \quad k \in \mathbb{Z} \quad , \\
(e_k \otimes \hat{u}(k))(t) &:= e_k(t) \hat{u}(k), \quad t \in [0,1] \quad .
\end{aligned}$$

As is well-known, the Banach-valued version of Fejer's theorem holds, that is, the sequence of Cesaro means of the sequence $\{F_N u\}_{N=0}^{\infty}$ converges to u in $L^p(0,1;X)$ for every $u \in L^p(0,1;X)$ and every $1 \leq p < \infty$.

It follows in particular that the vector space of trigonometric polynomials

$$T(X) := \operatorname*{span}_{k\in\mathbb{Z}}\{e_k\}$$

is dense in $L^p(0,1;X)$, $p \geq 1$. It is also known that the sequence $\{F_N u\}_{N=0}^{\infty}$ converges to u in $L^p(0,1;X)$ for every $u \in L^p(0,1;X)$ iff the Riesz projection P defined on $T(X)$ by

$$Pu := \sum_{k\geq 0} e_k \otimes \hat{u}(k)$$

is bounded in the $L^p(0,1;X)$ norm.

This is the case when $1 < p < \infty$ and $X = \mathbb{C}$ [11] or more generally iff X has the UMD property.

Under these conditions the sequence of bounded projections $\{E_k\}_{k=0}^{\infty}$ in $L^p(0,1;X)$ defined by $E_0 := F_0$ and $E_k := F_k - F_{k-1}$, $k > 1$ is a Schauder decomposition of the space $L^p(0,1;X)$. When $X = \mathbb{C}$, this decomposition is unconditional iff $p = 2$. In remarkable papers Paley and Littlewood [10], [11] showed that the dyadic blocking of $\{E_k\}_{k=0}^{\infty}$ defined by $D_k := F_{2^k} - F_{2^{k-1}}$, $k = 1, 2, \ldots$, $D_0 := F_1$, is unconditional. This property has been extended to the case X is UMD by Bourgain [3]. A detailed proof of this fact can be found in Venni [12]. A careful analysis of this proof shows that the notion of R-boundedness (which is not mentioned explicitly) plays an important role. This has been the starting point of [4].

In view of Theorem 1.1 a sequence $\{L_k\}_{k=0}^{\infty}$ of bounded linear operators in $L^p(0,1;X)$, $1 < p < \infty$ satisfying (1.7) and (1.12) induces a bounded linear operator T_L satisfying (1.13). In particular condition (1.7) is satisfied when the operators $\{L_k\}_{k=0}^{\infty}$ are diagonal operators of the form:

$$
\begin{aligned}
L_o u &:= \sum_{l=-1}^{1} e_l \otimes M_l \hat{u}(l) \quad , \\
L_k u &:= \sum_{2^{k-1} < |l| \leq 2^k} e_l \otimes M_l \hat{u}(l) \quad , \quad k = 1, 2, \ldots \quad ,
\end{aligned}
$$

where $\{M_l\}_{l\in\mathbb{Z}}$ is a family of bounded operators in $L^p(0,1;X)$. In a recent work Arendt and Bu [1] found an interesting sufficient condition on the sequence $\{M_l\}_{l\in\mathbb{Z}}$ for the family $\{L_k\}_{k=0}^{\infty}$ to be R-bounded, namely

$$
R_M := R(\{M_l\}_{l\in\mathbb{Z}} \quad \text{and} \quad \{l(M_{l+1} - M_l); l \in \mathbb{Z}\}) < \infty \tag{2.1}
$$

This leads to the following

Theorem 2.1 *(Arendt-Bu)*

Let $u \in L^p(0,1;X)$, $1 < p < \infty$, X UMD space.

Let $\{\hat{u}(k)\}_{k\in\mathbb{Z}}$ be the sequence of Fourier-coefficients of u and let $\{M_l\}_{l\in\mathbb{Z}}$ be a sequence of bounded linear operators in X. Then if the condition (2.1) holds, then the sequence $\{M_k\hat{u}(k)\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of a (unique) function $v \in L^p(0,1;X)$ and there exists a constant $c > 0$ depending only on $p \in (1,\infty)$ and X such that

$$
||v||_{L^p(0,1;X)} \leq cR_M ||u||_{L^p(0,1;X)} \quad . \tag{2.2}
$$

Remark

Theorem 2.1 is a simplified version of a more involved Marcinkiewicz type theorem due to Štrkalj and Weis [13], which is a discrete version of the following operator-valued Mikhlin type theorem due to Weis [14].

Theorem 2.2 *(Weis '99)*

Let $1 < p < \infty$ and X be a UMD space.

Let $M \in C^1(\mathbb{R}\backslash\{0\}; \mathcal{L}(X))$. Then M is the symbol of a bounded operator in $L^p(\mathbb{R}; X)$ if the collections $\{M(\rho); \rho \in \mathbb{R}\backslash\{0\}\}$ and $\{\rho M'(\rho); \rho \in \mathbb{R}\backslash\{0\}\}$ are R-bounded in $\mathcal{L}(X)$.

Comments

This remarkable theorem is the first operator-valued multiplier theorem in $L^p(\mathbb{R}; X)$, $1 < p < \infty$ where X is <u>not</u> isomorphic to a Hilbert space. Another proof can be found in [5]. It should be mentioned that the content of [4] has been made available to Professor L. Weis in December '98.

3 Converse theorems, R-sectoriality and L_p-Maximal Regularity

In this section we shall present some results showing that the R-boundedness of the "multipliers" is a necessary condition and that this notion naturally leads to the notion of R-sectoriality.

Theorem 3.1 *(Weis '99) [14]*

Let $(X, \|\cdot\|)$ be a complex Banach space (not necessarily UMD) and let $M \in C^1(\mathbb{R}\backslash\{0\}; \mathcal{L}(X))$. If M is the symbol of a bounded operator in $L^p(\mathbb{R}; X)$, $1 < p < \infty$ then the family $\{M(\rho); \rho \in \mathbb{R}\backslash\{0\}\}$ is R-bounded.

Remarks

1. Another proof of this result can be found in [5].
2. An analogue of this result in the discrete case has been established by Arendt and Bu [1], namely if $\{M_l\}_{l\in\mathbb{Z}}$ is a "multiplier" in $L^p(0,1;X)$ then the family $\{M_l\}_{l\in\mathbb{Z}}$ is R-bounded in $\mathcal{L}(X)$.

An application of Theorem 3.1 leads to the following L_p-maximal regularity theorem.

Theorem 3.2 *Let $(X, \|\cdot\|)$ be a complex Banach space and let $A : D(A) \subset X \to X$ be a sectorial operator with spectral angle $\omega_A < \pi/2$, and let $1 \leq p < \infty$.*
If there exists a constant $M > 0$ such that

$$\|u'\|_{L^p(\mathbb{R};X)} + \|Au\|_{L^p(\mathbb{R};X)} \leq M\|u' + Au\|_{L^p(\mathbb{R};X)} \tag{3.1}$$

for every $u \in W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$, then the family

$$\{\rho(i\rho + A)^{-1}; \quad \rho \in \mathbb{R}\backslash\{0\}\} \tag{3.2}$$

is R-bounded.
Conversely if $1 < p < \infty$ and X is UMD, then condition (3.1) is also sufficient for (3.2) to hold.

Comments

1. The characterization of L_p-maximal regularity for abstract differential equations in a UMD space has been obtained independently by Kalton and Weis (see [14]).

2. The characterization in the case $X = L^q(\Omega;\mu),\ 1 \leq q < \infty$ has been presented in a seminar in Delft, December '98 by Professor L. Weis in terms of a square-function estimate:

   There exists a constant $C > 0$ such that for all $f \in L^q(L^2)$

   $$\|(\int_{\mathbb{R}} |tR(it,A)f(t)|^2 dt)^{1/2}\|_X \leq C\|(\int_{\mathbb{R}} |f(t)|^2 dt)^{1/2}\|_X \quad .$$

   This notion is in this special case equivalent to condition (3.2).

3. It is natural to call a sectorial operator A, R-sectorial if in the definition of sectoriality, the notion of uniform boundedness is replaced by R-boundedness. See [5], [14] for definitions and examples of R-sectorial operators and [8] for examples of sectorial operators which are <u>not</u> R-sectorial in $L^q(([0,1]),\ 1 < q < \infty,\ q \neq 2$.

4. As is shown in [1] Theorem 2.1 and its partial converse are strong enough to characterize L_p-maximal regularity in UMD spaces.

Concluding remarks

It appears that the notion of R-boundedness is an appropriate notion for operator-valued multiplier theorems. Applications of this notion has been made by B. de Pagter,

F. Sukochev and H. Witvliet to Schur type multipliers [6]. We recall that in Theorem 1.1, the R-boundedness of the family $\{L_k\}_{k=1}^{\infty}$ is in general only a sufficient condition for the boundedness of the operator T_L. Examples where the family $\{L_k\}_{k=0}^{\infty}$ in Theorem 1.1 is not R-bounded and the operator T_L is bounded can be found in [15].

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